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A Boundary Element Method for Predicting Wave Forces on Marine Bodies with Slow Yaw-Motion

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Abstract

Wave drift damping of floating bodies in slow yaw-motion is considered. There is no restrictions on the magnitude of the yaw-angle. The problem is formulated in the frame of reference rotating with the slow yaw-motion of the body, accounting for the Coriolis force. Potential theory is applied to describe the fluid flow. The problem is solved to leading order in the wave amplitude and the slow yaw-velocity by means of integral equations. The wave drift damping in the yaw-mode is obtained by means of conservation of angular momentum.

1 Introduction

Wave drift damping of floating bodies in the three horizontal modes of motion is a problem of current interest within offshore technology. Earlier works have considered theoretical and numerical predictions of wave drift damping due to translatory motions (Wichers and van Sluijs 1979, Huismans and Hermans 1985, Zhao et al. 1988, Nossen et al. 1991, Emmerhoff and Sclavounos 1992, Grue and Palm 1993). In this contribution a theory is presented for evaluating wave drift damping of a floating body in waves due to a slow yaw-motion, i.e. a slow rotation about the vertical axis. The theory is completely based on the derivations outlined by Grue and Palm (1994).

The problem is considered in the frame of reference rotating with the slow yaw-motion of the body. This frame of reference is connected to a fixed frame of reference by the rotation angle being a slowly varying function of time. No restrictions are made to the magnitude of the rotation angle. The angular velocity is assumed to be small compared to the wave frequency, however. In the rotating frame of reference the Coriolis force contributes to the equation of motion for the fluid, and thereby to the yaw-drift damping coefficient. The sentrifugal force and the fictive force due to the angular acceleration also appear in the equation of motion, but do not contribute to the wave forces to leading order in the yaw-velocity, however. Relevant to the present theory is a recent work by Newman (1993), who, different from our method, describes the problem in the absolute frame of reference and assumes that the rotation angle of the body is small.

Assuming incompressible fluid and neglecting viscous effects we may apply potential theory to describe the fluid motion. An exact expression is developed for the pressure in the rotating frame of reference. Next a set of boundary value problems are formulated for the potentials

governing the flow, which then are solved to leading order in the wave amplitude and the slow yaw-velocity by means of integral equations. The integral equations, where the unknown potentials appear as unknowns on the wetted body surface, are suitable for applying a panel method. A discretization of the free surface is needed for ordinary integration, however. The wave drift damping coefficient in the yaw-mode is found by means of conservation of angular momentum, where the final expression is on a form which may be used for numerical evaluation.

2 The equation of motion and the pressure

We consider a floating body performing a slow rotation about the vertical axis while being exposed to incoming waves. Two frames of reference are introduced, one absolute frame of reference, $O-x^0y^0z^0$, being fixed in space, and one relative frame of reference, $O-xyz$, following the slow rotation about the vertical axis. (x^0, y^0) and (x, y) denote horizontal coordinates, while z, z^0 denote coordinates along the vertical axis pointing upwards. $z = z^0$ coincides with the mean level of the free surface. Let α denote the rotation angle of the body and $\Omega = \dot{\alpha}$ the angular velocity, where a dot denotes time derivative. $O - x^0y^0z^0$ and $O - xyz$ are then related by

$$x = x^0 \cos\alpha + y^0 \sin\alpha \tag{1}$$
$$y = y^0 \cos\alpha - x^0 \sin\alpha \tag{2}$$
$$z = z^0 \tag{3}$$

Polar coordinates are introduced in the relative frame of reference, viz. $x = R\cos\theta$, $y = R\sin\theta$. The fluid motion is considered in the relative frame of reference. Neglecting viscous effects the equation of motion for the fluid then reads

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p - \rho g \nabla z + \mathbf{H} \tag{4}$$

where ρ denotes the fluid density, $\mathbf{v}$ the velocity in the relative frame of reference, p the pressure and g the acceleration due to gravity. The force $\mathbf{H}$ is composed by the Coriolis force, $-2\rho\boldsymbol{\Omega}\times\mathbf{v}$, the centrifugal force, $-\rho\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{x}$, and the fictive force due to the angular acceleration, $-\rho\dot{\boldsymbol{\Omega}}\times\boldsymbol{x}$, where $\boldsymbol{\Omega} = \Omega\mathbf{k}$, $\mathbf{k}$ the unit vector along the vertical coordinate. Thus,

$$\mathbf{H} = -2\rho\boldsymbol{\Omega}\times\mathbf{v} - \rho\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{x} - \rho\dot{\boldsymbol{\Omega}}\times\boldsymbol{x} \tag{5}$$

Let the velocity be decomposed by $\mathbf{v} = \mathbf{v}' - \boldsymbol{\Omega}\times\boldsymbol{x}$. Assuming that $\mathbf{v}'$ is irrotational we may obtain $\mathbf{v}'$ by the gradient of a velocity potential Φ', i.e. $\mathbf{v}' = \nabla\Phi'$. The equation of motion then gives

$$\nabla\left(\frac{\partial\Phi'}{\partial t} - \Omega\frac{\partial\Phi'}{\partial\theta} + \frac{1}{2}|\nabla\Phi'|^2\right) = \nabla\left(-\frac{p}{\rho} - gz\right) \tag{6}$$

By integration the following relation for the pressure is obtained, which is exact

$$-\frac{p}{\rho} = \frac{\partial\Phi'}{\partial t} - \Omega\frac{\partial\Phi'}{\partial\theta} + \frac{1}{2}|\nabla\Phi'|^2 + gz + C(t) \tag{7}$$

3 The boundary value problems

It is assumed that the angular velocity Ω is much smaller than the wave frequency ω, i.e.

$$\Omega/\omega << 1 \tag{8}$$

This means that the problem has a fast time scale with characteristic time $1/\omega$ and a slow time scale with characteristic time $1/\Omega$. In obtaining the yaw-drift damping coefficient a time average over the fast time scale is applied.

According to the assumptions above Φ' satisfies the Laplace equation. Φ' is then decomposed by

$$\Phi' = \phi_s + \Phi + \psi^{(2)} \tag{9}$$

where $\phi_s = \Omega\chi_6$ denotes the potential generated by the body when there are no waves, Φ the linear wave potential being proportional to the wave amplitude, and $\psi^{(2)}$ a time averaged second order potential being proportional to the wave amplitude squared.

3.1 The steady potential, χ_6

The free surface boundary condition reads

$$\frac{\partial \chi_6}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{10}$$

where we have neglected terms being $O(\Omega^2, \dot{\Omega})$. The body boundary condition reads

$$\frac{\partial \chi_6}{\partial n} = \mathbf{n} \cdot (\mathbf{k} \times \boldsymbol{x}) \equiv n_6 \quad \text{at the body} \tag{11}$$

In addition, $\partial\chi_6/\partial n = 0$ at the bottom of the fluid, and $\nabla\chi_6 \to 0$ in the far field.

3.2 The linear wave potential, Φ

By applying the individual derivative to the pressure at the free surface we obtain after linearization

$$\frac{\partial^2 \Phi}{\partial t^2} - 2\Omega\frac{\partial^2 \Phi}{\partial\theta\partial t} + 2\nabla_h\phi_s \cdot \nabla_h\frac{\partial \Phi}{\partial t} - \frac{\partial^2 \phi_s}{\partial z^2}\frac{\partial \Phi}{\partial t} + g\frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{12}$$

where ∇_h denotes the horizontal gradient. Furthermore, Φ satisfies $\partial\Phi/\partial n = 0$ at the bottom of the fluid. The part of Φ describing the the scattered and the radiated waves satisfies a radiation condition in the far field.

Introducing $\Phi = Re(\phi e^{i\omega t})$, where ω denotes the incoming wave frequency in the absolute frame of reference, (12) becomes

$$-K\phi - 2i\epsilon\frac{\partial \phi}{\partial \beta} - 2i\epsilon\frac{\partial \phi}{\partial \theta} + 2i\epsilon\nabla_h\chi_6 \cdot \nabla\phi_1 + i\phi\epsilon\nabla_h^2\chi_6 + \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{13}$$

where β denotes the wave angle of the incident waves in the relative frame of reference, $\epsilon \equiv \omega\Omega/g$, and we have neglected terms being $O(\Omega^2, \dot{\Omega})$.

We consider the diffraction problem, thus

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{at the body} \tag{14}$$

We next apply a perturbation procedure to the potential ϕ, i.e.

$$\phi = \phi^0 + \epsilon\phi^1 \tag{15}$$

where ϕ^0 satisfies

$$-K\phi^0 + \frac{\partial\phi^0}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{16}$$

$$\frac{\partial\phi^0}{\partial n} = 0 \quad \text{at the body} \tag{17}$$

ϕ^1 then satisfies

$$-K\phi^1 + \frac{\partial\phi^1}{\partial z} = 2i\frac{\partial\phi^0}{\partial\beta} + 2i\frac{\partial\phi^0}{\partial\theta} - 2i\nabla_h\chi_6\cdot\nabla_h\phi^0 - i\phi^0\nabla_h^2\chi_6 \quad \text{at} \quad z = 0 \tag{18}$$

$$\frac{\partial\phi^1}{\partial n} = 0 \quad \text{at the body} \tag{19}$$

It is convenient to decompose the problem for ϕ^1 further by introducing

$$\phi^1 = \phi^{11} + \phi^{12} + \phi^{13} \tag{20}$$

where ϕ^{11}, ϕ^{12}, ϕ^{13} satisfy the following set of boundary value problems

$$-K\phi^{11} + \frac{\partial\phi^{11}}{\partial z} = 2i\frac{\partial\phi^0}{\partial\beta} \quad \text{at} \quad z = 0 \tag{21}$$

$$\frac{\partial\phi^{11}}{\partial n} = 0 \quad \text{at the body} \tag{22}$$

$$-K\phi^{12} + \frac{\partial\phi^{12}}{\partial z} = 2i\frac{\partial\phi^0}{\partial\theta} \quad \text{at} \quad z = 0 \tag{23}$$

$$\phi^{12} \quad \text{not specified at the body} \tag{24}$$

$$-K\phi^{13} + \frac{\partial\phi^{13}}{\partial z} = -2i\nabla_h\chi_6\cdot\nabla_h\phi^0 - i\phi^0\nabla_h^2\chi_6 \quad \text{at} \quad z = 0 \tag{25}$$

$$\frac{\partial\phi^{13}}{\partial n} = -\frac{\partial\phi^{12}}{\partial n} \quad \text{at the body} \tag{26}$$

Both ϕ^{11} and ϕ^{12} may be obtained in terms of ϕ^0, i.e.

$$\phi^{11} = 2i\frac{\partial^2\phi^0}{\partial\beta\partial K}, \quad \phi^{12} = 2i\frac{\partial^2\phi^0}{\partial\theta\partial K} \tag{27}$$

3.3 The second order potential, $\psi^{(2)}$

The second order potential $\psi^{(2)}$ enters in the formulae for the second order fluid pressure and the drift moment only multiplied by the slow rotation velocity Ω. To first order in Ω it is then sufficient to consider the boundary value problem for $\psi^{(2)}$ when there is no rotation ($\Omega = 0$). The free surface boundary condition for $\psi^{(2)}$ then satisfies

$$\frac{\partial \psi^{(2)}}{\partial z} = -\frac{1}{g}\overline{\frac{\partial}{\partial t}\nabla\Phi\cdot\nabla\Phi} + \frac{1}{g^2}\overline{\frac{\partial \Phi}{\partial t}\frac{\partial^3\Phi}{\partial z\partial t^2}} + \frac{1}{g}\overline{\frac{\partial \Phi}{\partial t}\frac{\partial^2\Phi}{\partial z^2}} = -\frac{\omega}{2g}Im(\phi^0\frac{\partial^2\phi^{0*}}{\partial z^2}) \quad \text{at } z=0 \tag{28}$$

where a bar denotes time average and a star complex conjungate. In the diffraction problem $\partial\psi^{(2)}/\partial n = 0$ at the body surface. In addition, $\partial\psi^{(2)}/\partial n = 0$ at the bottom of the fluid, and $\nabla\psi^{(2)} \rightarrow 0$ in the far field.

4 Integral equations

The boundary value problems for the components of the wave potential are solved by means of integral equations. This is achieved by first introducing a Green function G^0 being a sink at $\boldsymbol{x} = \boldsymbol{\xi}$, satisfying the homogenous free surface boundary condition with no rotation, i.e.

$$-KG^0 + \frac{\partial G^0}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{29}$$

A function G^1 is then introduced which satisfies the following inhomogenous boundary condition at the free surface

$$-KG^1 + \frac{\partial G^1}{\partial z} = 2i\frac{\partial G^0}{\partial \theta} \quad \text{at} \quad z=0 \tag{30}$$

G^1 may be given very shortly by

$$G^1 = 2i\frac{\partial^2 G^0}{\partial\theta\partial K} \tag{31}$$

Green's theorem is first applied to G^0 and ϕ^0, giving

$$\int_{S_B}\phi^0\frac{\partial G^0}{\partial n}dS - 4\pi\phi_I = \begin{cases} -2\pi\phi^0(\boldsymbol{x}) & \boldsymbol{x}\in S_B \\ -4\pi\phi^0(\boldsymbol{x}) & \boldsymbol{x}\in\mathcal{V}\end{cases} \tag{32}$$

which determines ϕ^0. Here, S_B denotes the wetted body surface, $\mathcal{V}$ the fluid volume enclosed by S_B, the free surface, S_F, and the vertical circular cylinder, S_∞, with radius $R \rightarrow \infty$. n is pointing out of the fluid.

To obtain an integral equation for ϕ^1 we first apply Green's theorem to $\psi = \phi^{12} + \phi^{13}$ and G^0, giving

$$\int_{S_B}\psi\frac{\partial G^0}{\partial n}dS + \int_{S_F+S(R)}\left(\psi\frac{\partial G^0}{\partial n} - G^0\frac{\partial\psi}{\partial n}\right)dS = \begin{cases} -2\pi\psi(\boldsymbol{x}) & \boldsymbol{x}\in S_B \\ -4\pi\psi(\boldsymbol{x}) & \boldsymbol{x}\in\mathcal{V}\end{cases} \tag{33}$$

where $S(R)$ denotes the surface of a vertical circular cylinder with radius $R < \infty$ surrounding the body. Next Green's theorem is applied to ϕ^0 and G^1 giving

$$\int_{S_B}\phi^0\frac{\partial G^1}{\partial n}dS + \int_{S_F+S(R)}\left(\phi^0\frac{\partial G^1}{\partial n} - G^1\frac{\partial\phi^0}{\partial n}\right)dS = 0 \tag{34}$$

By subtracting (34) from (33) and using the boundary conditions it is shown in Grue and Palm (1994) that we obtain for ϕ^1:

$$\int_{S_B} \phi^1 \frac{\partial G^0}{\partial n} dS - \int_{S_B} \left(\phi^0 \frac{\partial G^1}{\partial n} - 2i \frac{\partial \phi^0}{\partial \beta} \frac{\partial^2 G^0}{\partial n \partial K} \right) dS$$

$$-2i \int_{S_F} \phi^0 (\nabla_h \chi_6 \cdot \nabla_h G^0 + \frac{1}{2} G^0 \nabla_h^2 \chi_6) dS = \begin{cases} -2\pi \phi^1(\boldsymbol{x}) & \boldsymbol{x} \in S_B \\ -4\pi \phi^1(\boldsymbol{x}) & \boldsymbol{x} \in \mathcal{V} \end{cases} \quad (35)$$

where the first case is an integral equation for ϕ^1.

5 The damping moment

Let us then evaluate the moment about the vertical axis, M_z. We start with considering the vector product between the coordinate $\boldsymbol{x}$ and the equation of motion (4). By integrating over the fluid volume, using Gauss' theorem and the transport theorem, we obtain

$$M_z = -\rho \frac{d}{dt} \int_{\mathcal{V}} \mathbf{k} \cdot (\boldsymbol{x} \times \mathbf{v}') d\mathcal{V} - \rho \int_{S_\infty} v'_\theta v'_n R dS \quad (36)$$

where we have applied that that the control surface is a vertical circular cylinder with axis passing through origin. It may be shown that

$$\overline{\int_{\mathcal{V}} \mathbf{k} \cdot (\boldsymbol{x} \times \mathbf{v}') d\mathcal{V}} = -\frac{\omega}{2g} \int_{S_F} [\chi_6 Im(\phi^0 \phi_{zz}^{0*}) + Im(\phi_\theta^0 \phi^{0*})] dS \quad (37)$$

where we have applied the boundary condition (28) for $\partial \psi^{(2)} / \partial z$ at $z = 0$. We then obtain for the time averaged yaw-moment

$$\overline{M_z} = M_{z0} + \epsilon \rho B_{66} \quad (38)$$

where M_{z0} denotes the yaw-moment when there is no rotation ($\Omega = 0$), and B_{66} denotes the damping coefficient due to the slow yaw-velocity, given by

$$B_{66} = -\frac{1}{2} \frac{\partial}{\partial \beta} \int_{S_F} \left[\chi_6 Im(\phi^0 \phi_{zz}^{0*}) + Im(\phi_\theta^0 \phi^{0*}) \right] dS - \frac{1}{2} \int_{S_\infty} Re \left[\phi_\theta^0 \phi_R^{1*} + \phi_\theta^1 \phi_R^{0*} \right] dS \quad (39)$$

We may develop this expression further to obtain

$$B_{66} = -\frac{\partial}{\partial K} Im \int_{S_\infty} \phi_\theta^0 \phi_{\theta R}^{0*} dS + \frac{1}{2} Re \int_{S_\infty} \left[\phi_\theta^0 \phi_R^{13*} + \phi_\theta^{13} \phi_R^{0*} \right] dS$$

$$-Im \int_{S_B} (\nabla \phi_{\beta K}^{0*} \cdot \nabla \phi^0) n_6 dS - Im \int_{C_B} \phi_{\beta K}^{0*} K \phi^0 n_6 dl$$

$$- \int_{S_F} Im[\frac{1}{2} \phi^0 \phi_\beta^{0*} \chi_{6zz} - \phi_\beta^{0*} \nabla_h \phi^0 \cdot \nabla_h \chi_6] dS \quad (40)$$

where C_B denotes the waterline of the body.

6 Discussion

A theory for evaluating the yaw-drift damping coefficient B_{66} due to a small yaw-velocity of a floating body is outlined. The derived formulae may be applied directly for obtaining numerical values of B_{66}. The first step is to evaluate the potentials ϕ^0 and ϕ^1 which are

determined by the integral equations (32) and (35), respectively. These integral equations are suitable for numerical solution by means of a panel method. As eq. (40), shows it is sufficient to know ϕ^0 and the far field behaviour of ϕ^{13} to obtain B_{66}. One alternative is then to find ϕ^{13} by $\phi^1 - \phi^{11} - \phi^{12}$, where ϕ^1 is given by (35) and ϕ^{11} and ϕ^{12} by (27). A different procedure is to solve the boundary value problem (25) - (26) to obtain ϕ^{13}.

The formula for B_{66} is obtained by applying conservation of angular momentum, where the final expression (40) contains integrals over the body surface, the water line of the body, the free surface, and a control surface at infinity. All the terms of (40) are convergent integrals. The free surface integral converges rapidly since χ_{6zz} and $\nabla\chi_6$ decay fast for $R \to \infty$. The integrals over the far field control surface may easily be evaluated by applying the far field properties of ϕ^0 and ϕ^{13}.

B_{66} may alternatively be found by integrating the pressure over the body surface. This method requires, however, that ϕ^0, ϕ^1, $\nabla\phi^0$ and $\nabla\phi^1$ are evaluated at the body surface. If a low-order panel method is applied as numerical tool, ϕ^0 and ϕ^1 are then most suitably obtained by source distributions, instead of the integral equations (32) and (35), which are obtained by means of Greens theorem.

The components of the horizontal damping force, B_{16}, B_{26}, due to the slow rotation of the body may be obtaind by conservation of linear momentum, see Grue and Palm (1994).

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